The 3-wave PDEs for resonantly interacting triads

Ruth Martin & Harvey Segur

Waves and singularities in incompressible fluids

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What are the 3-wave equations? What is a resonant triad?

Let $\eta(\vec{X},T)$ denote the elevation of the ocean's surface, in the presence of three trains of dispersive waves. With $0 < \varepsilon << 1$,

$$\eta(\vec{X},T) = \varepsilon \sum_{j=1}^{3} A_j(\varepsilon\vec{X},\varepsilon T) \ \exp\{i\vec{k}_j \cdot \vec{X} - i\omega_j T\} + (c\ c).$$

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Then the three, *complex-valued* wave envelopes can exchange energy according to

$$\partial_t A_j(x,t) + \vec{c}_j \cdot \nabla A_j(x,t) = \zeta_j \cdot A_k^*(x,t) \cdot A_l^*(x,t),$$

where $\vec{c}_j$ is j^{th} group velocity, ζ_j is real-valued interaction coefficient, and $j,k,l = 1,2,3$, cyclically.

$$\partial_t A_j(x,t) + \vec{c}_j \cdot \nabla A_j(x,t) = \zeta_j \cdot A_k^*(x,t) \cdot A_l^*(x,t)$$

Comments:

– This model describes the simplest possible nonlinear interaction among dispersive wave trains.

– The model admits no dissipation.

– These are "envelope equations", like NLS.

– These are **not** equivalent to the "3-wave equations" that Toan Nguyen discussed last Tuesday.

– By suitable rescaling, the interaction coefficients $\{\zeta_j\}$ can be written as real-valued (as here) or as pure imaginary, or as $\{+1/-1\}$.

Which physical problems admit resonant triads?

- Gravity-driven water waves, without surface tension?
- Capillary water waves, without gravity?
- Capillary-gravity waves?
- Internal waves in a stratified ocean?
- Electromagnetic waves in a dielectric medium?
  – in a χ_2 material?
  – in a χ_3 material?
- Laser pointers?

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  - – in a χ_2 material? Yes
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This question is answered by a simple test of the dispersion relation of the linearized problem.

$$\partial_t A_j(x,t) + \vec{c}_j \cdot \nabla A_j(x,t) = \zeta_j \cdot A_k^*(x,t) \cdot A_l^*(x,t)$$

Properties of this system of equations

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Mathematical status of these equations

$$\partial_t A_j(x,t) + \vec{c}_j \cdot \nabla A_j(x,t) = \zeta_j \cdot A_k^*(x,t) \cdot A_l^*(x,t)$$

(1) If all three wavetrains have spatially uniform envelopes, then $\vec{c}_j \cdot \nabla A_j(x,t) = 0,$ and the 3 PDEs reduce to 3 complex ODEs:

$$\frac{d(A_j)}{dt} = \zeta_j A_k^* A_l^*.$$

Bretherton (1964) found 3 conservation laws, and built the general solution of the equations explicitly in terms of elliptic functions.

(2) Zakharov & Manakov (1973) found a Lax pair for the PDEs, then Zakharov & Manakov (1976) and Kaup (1976) solved the PDEs in unbounded 3-D space.

(3) Nothing is known about the solution of the PDEs on a finite interval, with periodic or any other boundary conditions.

Our objective:
Construct the general solution of the 3-wave PDEs

Q: What does "general solution of a PDE" mean?

- The general solution of an N^{th} order system of ordinary differential equations is a set of functions (or a single function) that solve the ODE(s) and that admit exactly N free constants (which can be viewed as N constants of integration, or as N pieces of initial data).
- *Proposal*: Given a system of N partial differential equations that are evolutionary in time, we define its general solution to be a set of functions (or a single function) that solve the PDEs, and that admit N arbitrary *functions* that are independent of the PDEs, but might also be required to satisfy conditions external to the PDEs, like sufficient differentiability.

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Q: An example of a PDE for which a general solution is known?

A: D'Alembert's solution of the wave equation in 1D:

$$u(x,t) = f(x-ct) + g(x+ct).$$

f(•) and g(•) must be twice-differentiable. No other constraints.

Q: Might the 3-wave PDEs provide a more complicated example?

Step 1: Solve the 3-wave ODEs

The 3-wave ODEs are three coupled, complex-valued ODEs of the form

$$\frac{d(A_j)}{dt} = \zeta_j A_k^* A_l^*,$$

where each of the three interaction coefficients, ζ_j , is a specified real number. These ODEs are equivalent to six real-valued ODEs, so any solution of the ODEs necessarily resides in a six-dimensional phase space.

We show below that these coupled ODEs are Hamiltonian, so the general solution can be specified in terms of three sets of *action-angle variables*.

ODEs: action-angle variables

Not all ODEs admit action-angle variables, but those that do are necessarily completely integrable. Action-angle variables have a nice geometric interpretation. For the 3-wave ODEs, the solution necessarily resides on a three-dimensional manifold within a six-dimensional phase space. Each action variables is a constant of the motion, and these three constants define the three-dimensional manifold in question. Then the three angle-variables define the trajectory of the solution on this manifold.

From the ODEs, observe that

$$\frac{d(A_1)}{dt} = \zeta_1 A_2^* A_3^*, \quad \frac{d(A_1^*)}{dt} = \zeta_1 A_2 A_3 .$$

ODEs: the action variables

Cross-multiply and add to obtain

$$\frac{d}{dt}(A_1 A_1^*) = \zeta_1 \{A_1^* A_2^* A_3^* + A_1 A_2 A_3\}.$$

$$\Rightarrow \quad K_1 = \frac{|A_1|^2}{\zeta_1} - \frac{|A_3|^2}{\zeta_3}$$

is a constant of the motion, and so is

$$K_2 = \frac{|A_2|^2}{\zeta_2} - \frac{|A_3|^2}{\zeta_3}.$$

ODEs: the action variables

All three of $K_1 = \dfrac{|A_1|^2}{\zeta_1} - \dfrac{|A_3|^2}{\zeta_3}, \quad K_2 = \dfrac{|A_2|^2}{\zeta_2} - \dfrac{|A_3|^2}{\zeta_3}, \quad K_1 - K_2$

are constants of the motion.

– If any two of $\{\zeta_1, \zeta_2, \zeta_3\}$ have *different signs*, then one of $\{K_1, K_2, K_1 - K_2\}$ guarantees that the solutions are bounded, for all time –this is the *non-explosive* case.

– If all three of $\{\zeta_1, \zeta_2, \zeta_3\}$ have the *same sign*, then none of $\{K_1, K_2, K_1 - K_2\}$ bounds the solutions, so all three wavetrains can blow up in finite time – this is the *explosive* case, the focus of today's work.

ODEs: the action variables

The third constant of the motion is

$$H = i\left\{A_1^* A_2^* A_3^* - A_1 A_2 A_3\right\}$$

Note that the complex conjugate of H is itself, so H is real-valued. H is also the Hamiltonian of the system with with 3 pairs of conjugate variables:

$$\{A_1, A_1^*\}, \quad \{A_2, A_2^*\}, \quad \{A_3 A_3^*\}.$$

The three *action variables* for this set of ODEs are algebraic combinations of (K_1, K_2, H), so these three constants of the motion define the three-dimensional manifold on which the solution lives.

ODEs: the action variables

In the explosive case, all of the interaction coefficients have the same sign (σ), so rescale $\{A_1, A_2, A_3, t)$ according to

$$t = T\tau, \qquad A_1(x,t) = \frac{1}{\sqrt{|\zeta_2\zeta_3|T}}a_1(x,\tau),$$

$$A_2(x,t) = \frac{1}{\sqrt{|\zeta_3\zeta_1|T}}a_2(x,\tau), \quad A_3(x,t) = \frac{1}{\sqrt{|\zeta_1\zeta_2|T}}a_3(x,\tau).$$

Then the three constants of the motion become

$$\tilde{K}_1 = (\left|\zeta_1\zeta_2\zeta_3\right|T^2)^{-1}\left\{\left|a_1\right|^2 - \left|a_3\right|^2\right\}, \quad \tilde{K}_2 = (\left|\zeta_1\zeta_2\zeta_3\right|T^2)^{-1}\left\{\left|a_2\right|^2 - \left|a_3\right|^2\right\},$$

$$i\tilde{H} = (\left|\zeta_1\zeta_2\zeta_3\right|T^2)^{-1}(a_1^* a_1^* a_1^* - a_1 a_2 a_3).$$

ODEs: the angle variables

To find the corresponding angle variables, it is convenient to construct a formal Laurent series of the solution in the neighborhood of a pole of order 1:

$$A_j(t) = \frac{\rho_j e^{i\theta_j}}{(t-t_0)}[1+\alpha_j(t-t_0)+\beta_j(t-t_0)^2+\gamma_j(t-t_0)^3$$

$$+\delta_j(t-t_0)^4+\varepsilon(t-t_0)^5+\ldots].$$

For each j, $\{\rho_j, \theta_j, t_0\}$ are real-valued constants, with $\rho_j > 0$, while $\{\alpha_j, \beta_j, \gamma_j, \delta_j, \ldots\}$ are complex-valued constants. Insert this form into the ODEs and solve, order by order for the unknown coefficients.

ODEs: the angle variables

At leading order, the representative ODE becomes

$$\frac{\rho_1}{(t-t_0)^2}[-1+\ldots] = \sigma\cdot\exp\{i(\theta_1+\theta_2+\theta_3)\frac{\rho_2\rho_3}{(t-t_0)^2}[1+\ldots].$$

The ρ_j are necessarily positive, so we need

$$\sigma\cdot\exp\{i(\theta_1+\theta_2+\theta_3) = -1$$

But there are no other constraints on the real numbers $(\theta_1,\ \theta_2,\ \theta_3)$

so we may choose any two of $(\theta_1,\ \theta_2,\ \theta_3)$

ODEs: angle variables

$$A_j(t) = \frac{\rho_j e^{i\theta_j}}{(t-t_0)}[1+\alpha_j(t-t_0)+\beta_j(t-t_0)^2+\gamma_j(t-t_0)^3$$

$$+\delta_j(t-t_0)^4+\varepsilon(t-t_0)^5+\ldots].$$

In addition to two of $(\theta_1,\ \theta_2,\ \theta_3)$, the last angle variable is (t_0).
All of the angle variables are obtained at leading order.

ODEs: higher order terms in series

$$A_j(t) = \frac{\rho_j e^{i\theta_j}}{(t - t_0)}[1 + \alpha_j(t - t_0) + \beta_j(t - t_0)^2 + \gamma_j(t - t_0)^3$$

$$+\delta_j(t - t_0)^4 + \varepsilon(t - t_0)^5 + \ldots].$$

The higher order coefficients in the series are $\{\alpha_{j,}\ \beta_{j,}\ \ldots.\}$. These are obtained by solving coupled linear algebraic equations at each integer power of $(t - t_0)$. Typically, these algebraic equations have non-homogeneous terms, coming from previously found coefficients in the series. These forcing terms can be complex, so one actually needs to solve six coupled algebraic equations at each order.

ODEs: singular points

$$A_j(t) = \frac{\rho_j e^{i\theta_j}}{(t-t_0)}[1+\alpha_j(t-t_0)+\beta_j(t-t_0)^2+\gamma_j(t-t_0)^3$$

$$+\delta_j(t-t_0)^4+\varepsilon(t-t_0)^5+\ldots].$$

Because each new order involves a higher power of $(t - t_0)$ the coefficient matrix changes in a predictable way. For $(\alpha_{1,}\ \alpha_2, \alpha_3)$ coefficient matrices are nonsingular, so $(\alpha_{1,}\ \alpha_2, \alpha_3)$ are determined uniquely, and they are all zero.

For $\{\beta_{\mathrm{j}}\}$, one coefficient matrix is singular, so the real parts of any two of $\{\beta_1, \beta_2, \beta_3\}$can be chosen at will. One finds that the two free choices of $\{\beta_1, \beta_2, \beta_3\}$ are not determined by the ODEs directly, but they turn out to be algebraic combinations of the Manley-Rowe constants.

ODEs: the rest of the series

$$A_j(t) = \frac{\rho_j e^{i\theta_j}}{(t-t_0)}[1+\alpha_j(t-t_0)+\beta_j(t-t_0)^2+\gamma_j(t-t_0)^3$$

$$+\delta_j(t-t_0)^4+\varepsilon(t-t_0)^5+\ldots].$$

Similarly, the coefficient matrix for the real part of $\{\gamma_1, \gamma_2, \gamma_3\}$ is not singular, but the one for the imaginary parts of $\{\gamma_1, \gamma_2, \gamma_3\}$ is singular. In this case, the imaginary parts of γ_j are proportional to the Hamiltonian, *H*.

After $(t - t_0)^3$, there are no more singular coefficient matrices: every coefficients is uniquely determined, in terms of earlier terms in the series. Convergence of the series is guaranteed because the solution is comprised of elliptic functions.

PDEs (finally!)

Consider next the PDE version of three-wave equations:

$$\partial_t A_j(x,t) + \vec{c}_j \cdot \nabla A_j(x,t) = \zeta_j \cdot A_k^*(x,t) \cdot A_l^*(x,t)$$

Now spatial derivative is involved, so the analysis is more involved. First, we summarize the analysis of Martin & Segur (2015), then we consider the full problem.

Recall that for the ODEs, the action-angle variables are $\{K_1, K_2, H\}$, and $\{\theta_1, \theta_2,\ t_0\}$. The basic hypothesis of this work is that one might be able to replace each of the six arbitrary constants in the solution of the ODEs with six arbitrary functions of x, and obtain in this way a large family of solutions of the PDEs.

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The role played by t_o is more fundamental than that of the other variables, so M&S (2016) replaced five of the six free constants in the solution of ODEs with five free functions, allowing t_o to remain a constant. The short summary of that work is that everything works, mostly as it does for the ODEs. The new free functions of x must be infinitely differentiable, because they get differentiated over and over as one goes to higher and higher terms in the Laurent series. And higher derivatives must be bounded in terms of lower derivatives.

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M & S (2016) required that for each free function, $f(x)$, there must be a finite, positive number, k, such that higher derivatives satisfy

$$\left| \frac{d^n f(x)}{dx^n} \right| \leq |k|^n \, |f(x)|.$$

With this constraint, they proved convergence of the Laurent series, with radii of convergence that were only nominally smaller than the distance to the next pole.

PDEs (finally!)

$$\partial_t A_j(x,t) + \vec{c}_j \cdot \nabla A_j(x,t) = \zeta_j \cdot A_k^*(x,t) \cdot A_l^*(x,t)$$

The final stage of work on this problem involves replacing the last free parameter, t_0 , with a free function of x, in order to obtain the full "general solution" of the PDEs, with six arbitrary functions of x, subject only to mild constraints. The results obtained so far have been obtained at ICERM, during the semester-long program that is now coming to an end. The results so far are promising, but it would be premature to forecast the final outcome at this time.

If everything works, then this set of PDEs will join the linear wave equation in one dimension as one of the very few PDEs for which a general solution is available.

Thank you for your attention.